CHAPTER 7

Trajectory generation

7.1 INTRODUCTION

In this chapter, we concern ourselves with methods of computing a trajectory that describes the desired motion of a manipulator in multidimensional space. Here, **trajectory** refers to a time history of position, velocity, and acceleration for each degree of freedom.

This problem includes the human-interface problem of how we wish to *specify* a trajectory or path through space. In order to make the description of manipulator motion easy for a human user of a robot system, the user should not be required to write down complicated functions of space and time to specify the task. Rather, we must allow the capability of specifying trajectories with simple descriptions of the desired motion, and let the system figure out the details. For example, the user might want to be able to specify nothing more than the desired goal position and orientation of the end-effector and leave it to the system to decide on the exact shape of the path to get there, the duration, the velocity profile, and other details.

We also are concerned with how trajectories are *represented* in the computer after they have been planned. Finally, there is the problem of actually computing the trajectory from the internal representation—or *generating* the trajectory. Generation occurs at *run time*; in the most general case, position, velocity, and acceleration are computed. These trajectories are computed on digital computers, so the trajectory points are computed at a certain rate, called the **path-update rate.** In typical manipulator systems, this rate lies between 60 and 2000 Hz.

7.2 GENERAL CONSIDERATIONS IN PATH DESCRIPTION AND GENERATION

For the most part, we will consider motions of a manipulator as motions of the tool frame, $\{T\}$, relative to the station frame, $\{S\}$. This is the same manner in which an eventual user of the system would think, and designing a path description and generation system in these terms will result in a few important advantages.

When we specify paths as motions of the tool frame relative to the station frame, we decouple the motion description from any particular robot, end-effector, or workpieces. This results in a certain modularity and would allow the same path description to be used with a different manipulator—or with the same manipulator, but a different tool size. Further, we can specify and plan motions relative to a moving workstation (perhaps a conveyor belt) by planning motions relative to the station frame as always and, at run time, causing the definition of $\{S\}$ to be changing with time.

As shown in Fig. 7.1, the basic problem is to move the manipulator from an initial position to some desired final position—that is, we wish to move the tool frame from its current value, $\{T_{initial}\}$, to a desired final value, $\{T_{final}\}$. Note that, in general, this motion involves both a change in orientation and a change in the position of the tool relative to the station.

Sometimes it is necessary to specify the motion in much more detail than by simply stating the desired final configuration. One way to include more detail in a path description is to give a sequence of desired **via points** (intermediate points between the initial and final positions). Thus, in completing the motion, the tool frame must pass through a set of intermediate positions and orientations as described by the via points. Each of these via points is actually a frame that specifies both the position and orientation of the tool relative to the station. The name **path points** includes all the via points plus the initial and final points. Remember that, although we generally use the term "points," these are actually frames, which give both position and orientation. Along with these *spatial* constraints on the motion, the user could also wish to specify *temporal* attributes of the motion. For example, the time elapsed between via points might be specified in the description of the path.

Usually, it is desirable for the motion of the manipulator to be *smooth*. For our purposes, we will define a smooth function as a function that is continuous and has a continuous first derivative. Sometimes a continuous second derivative is also

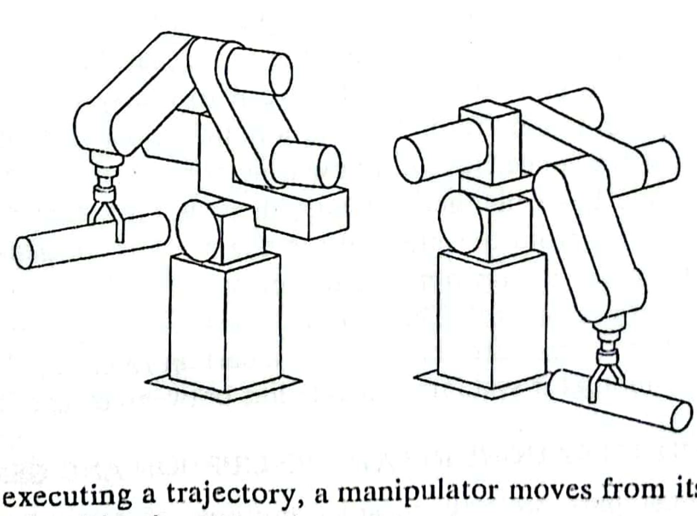


FIGURE 7.1: In executing a trajectory, a manipulator moves from its initial position to a desired goal position in a smooth manner.

desirable. Rough, jerky motions tend to cause increased wear on the mechanism and cause vibrations by exciting resonances in the manipulator. In order to guarantee smooth paths, we must put some sort of constraints on the spatial and temporal qualities of the path *between* the via points.

At this point, there are many choices that may be made and, consequently, a great variety in the ways that paths might be specified and planned. Any smooth functions of time that pass through the via points could be used to specify the exact path shape. In this chapter, we will discuss a couple of simple choices for these functions. Other approaches can be found in [1, 2] and [13–16].

7.3 JOINT-SPACE SCHEMES

In this section, we consider methods of path generation in which the path shapes (in space and in time) are described in terms of functions of joint angles.

Each path point is usually specified in terms of a desired position and orientation of the tool frame, $\{T\}$, relative to the station frame, $\{S\}$. Each of these via points is "converted" into a set of desired joint angles by application of the inverse kinematics. Then a smooth function is found for each of the n joints that pass through the via points and end at the goal point. The time required for each segment is the same for each joint so that all joints will reach the via point at the same time, thus resulting in the desired Cartesian position of $\{T\}$ at each via point. Other than specifying the same duration for each joint, the determination of the desired joint angle function for a particular joint does not depend on the functions for the other joints.

Hence, joint-space schemes achieve the desired position and orientation at the via points. In between via points, the shape of the path, although rather simple in joint space, is complex if described in Cartesian space. Joint-space schemes are usually the easiest to compute, and, because we make no continuous correspondence between joint space and Cartesian space, there is essentially no problem with singularities of the mechanism.

Cubic polynomials

Consider the problem of moving the tool from its initial position to a goal position in a certain amount of time. Inverse kinematics allow the set of joint angles that correspond to the goal position and orientation to be calculated. The initial position of the manipulator is also known in the form of a set of joint angles. What is required is a function for each joint whose value at t_0 is the initial position of the joint and whose value at t_f is the desired goal position of that joint. As shown in Fig. 7.2, there are many smooth functions, $\theta(t)$, that might be used to interpolate the joint value.

In making a single smooth motion, at least four constraints on $\theta(t)$ are evident. Two constraints on the function's value come from the selection of initial and final values:

$$\begin{aligned} \theta(0) &= \theta_0, \\ \theta(t_f) &= \theta_f. \end{aligned} \tag{7.1}$$

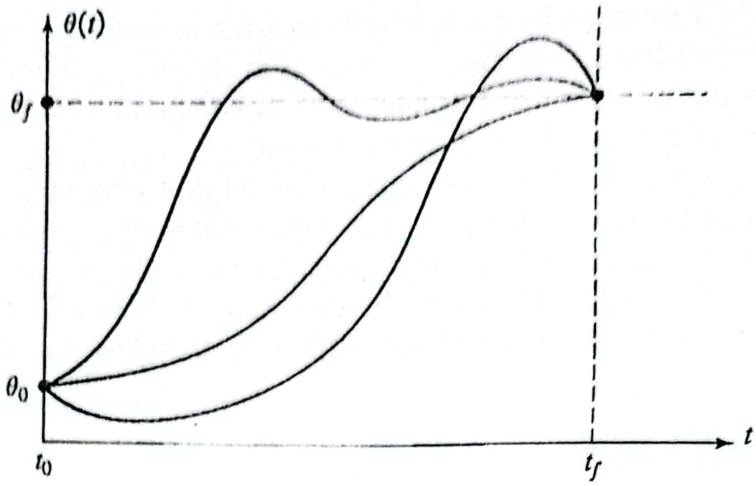


FIGURE 7.2: Several possible path shapes for a single joint.

An additional two constraints are that the function be continuous in velocity, which in this case means that the initial and final velocity are zero:

$$\dot{\theta}(0) = 0,$$
$$\dot{\theta}(t_f) = 0. \tag{7.2}$$

These four constraints can be satisfied by a polynomial of at least third degree. (A cubic polynomial has four coefficients, so it can be made to satisfy the four constraints given by (7.1) and (7.2).) These constraints uniquely specify a particular cubic. A cubic has the form

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \tag{7.3}$$

so the joint velocity and acceleration along this path are clearly

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2,$$
$$\ddot{\theta}(t) = 2a_2 + 6a_3 t. \tag{7.4}$$

Combining (7.3) and (7.4) with the four desired constraints yields four equations in four unknowns:

$$\theta_0 = a_0,$$
$$\theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3,$$
$$0 = a_1, \tag{7.5}$$
$$0 = a_1 + 2a_2 t_f + 3a_3 t_f^2.$$

Solving these equations for the a_i, we obtain

$$a_0 = \theta_0,$$
$$a_1 = 0,$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0), \tag{7.6}$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0).$$

Using (7.6), we can calculate the cubic polynomial that connects any initial joint-angle position with any desired final position. This solution is for the case when the joint starts and finishes at zero velocity.

EXAMPLE 7.1

A single-link robot with a rotary joint is motionless at $\theta = 15$ degrees. It is desired to move the joint in a smooth manner to $\theta = 75$ degrees in 3 seconds. Find the coefficients of a cubic that accomplishes this motion and brings the manipulator to rest at the goal. Plot the position, velocity, and acceleration of the joint as a function of time.

Plugging into (7.6), we find that

$$\begin{aligned} a_0 &= 15.0, \\ a_1 &= 0.0, \\ a_2 &= 20.0, \\ a_3 &= -4.44. \end{aligned} \tag{7.7}$$

Using (7.3) and (7.4), we obtain

$$\begin{aligned} \theta(t) &= 15.0 + 20.0t^2 - 4.44t^3, \\ \dot{\theta}(t) &= 40.0t - 13.33t^2, \\ \ddot{\theta}(t) &= 40.0 - 26.66t. \end{aligned} \tag{7.8}$$

Figure 7.3 shows the position, velocity, and acceleration functions for this motion sampled at 40 Hz. Note that the velocity profile for any cubic function is a parabola and that the acceleration profile is linear.

Cubic polynomials for a path with via points

So far, we have considered motions described by a desired duration and a final goal point. In general, we wish to allow paths to be specified that include intermediate via points. If the manipulator is to come to rest at each via point, then we can use the cubic solution of Section 7.3.

Usually, we wish to be able to pass through a via point without stopping, and so we need to generalize the way in which we fit cubics to the path constraints.

As in the case of a single goal point, each via point is usually specified in terms of a desired position and orientation of the tool frame relative to the station frame. Each of these via points is "converted" into a set of desired joint angles by application of the inverse kinematics. We then consider the problem of computing cubics that connect the via-point values for each joint together in a smooth way.

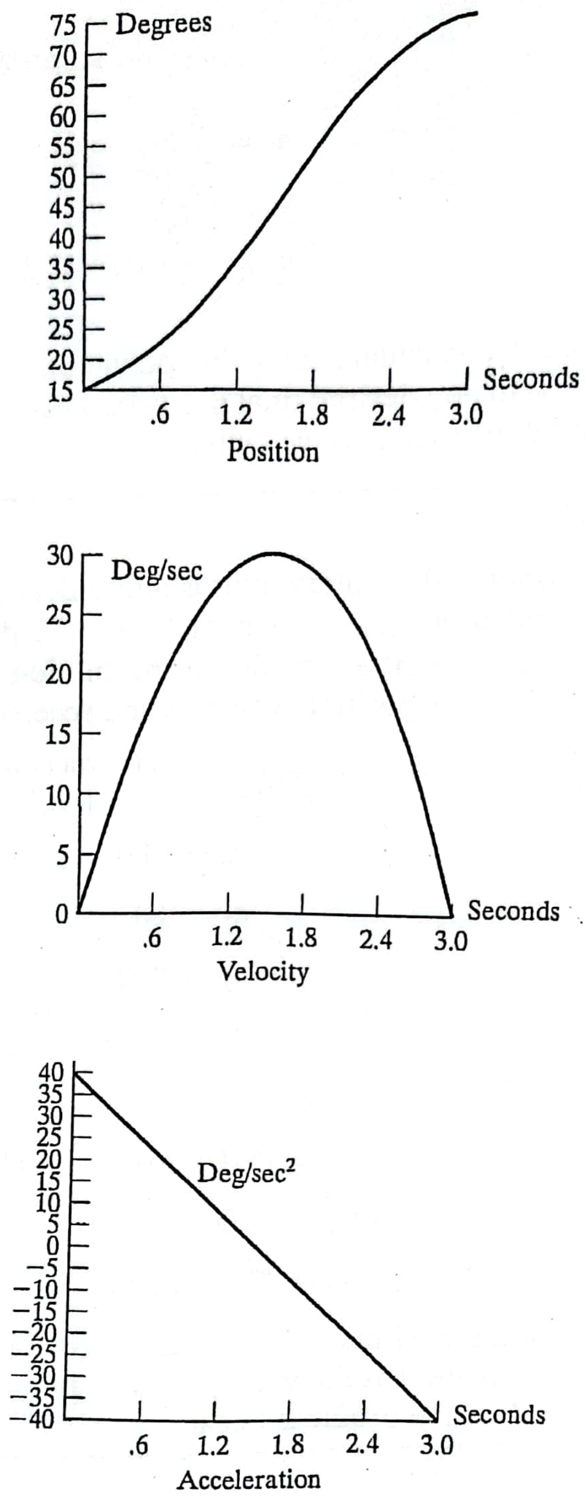


FIGURE 7.3: Position, velocity, and acceleration profiles for a single cubic segment that starts and ends at rest.

If desired velocities of the joints at the via points are known, then we can construct cubic polynomials as before; now, however, the velocity constraints at each end are not zero, but rather, some known velocity. The constraints of (7.3) become

$$\dot{\theta}(0) = \dot{\theta}_0,$$
$$\dot{\theta}(t_f) = \dot{\theta}_f. \tag{7.9}$$

only limited axis movement. The addition of a wrist at the end of the robot's arm extends the mobility of the robotic system. The addition of the wrist also increases the dimensions of the work envelope.

Figure 2–6 illustrates the movement and the axes of a wrist. With the majority of wrists in use today, an additional two or three axes are added to the robot's mobility. Figure 2–6 shows a wrist that develops three additional axes of movement: the yaw axis, the pitch axis, and the roll axis.

The yaw axis describes the wrist's angular movement from the left side to the right side. This motion can range from a 90° movement to a 270° movement, depending on the design of the wrist.

The pitch axis describes the wrist's rotational movement up and down. The angular motion of the pitch can range from merely a few degrees of motion to 270°, depending on the application of the wrist.

The roll axis describes the rotation around the end of the wrist. The roll axis can provide rotation up to 360°. With an end effector connected to the roll axis, a full 360° of rotation can be achieved.

The addition of these extra axes allows the robotic system to be very flexible. The degrees of rotation that the wrist provides are variable. For example, the yaw can have 270° of travel, the pitch can allow 90° to 110° of travel, and the roll, as stated earlier, can add 360° of rotation. Or a wrist can develop two rolls and no pitch. But however the wrist axes are designed, the addition of the wrist to the robot's arm allows the end effector to reach into areas that could not be reached by robots using only one of the four coordinate systems for the arm. The flexibility of the system is thus increased with the different wrist designs used in the robotic operation.

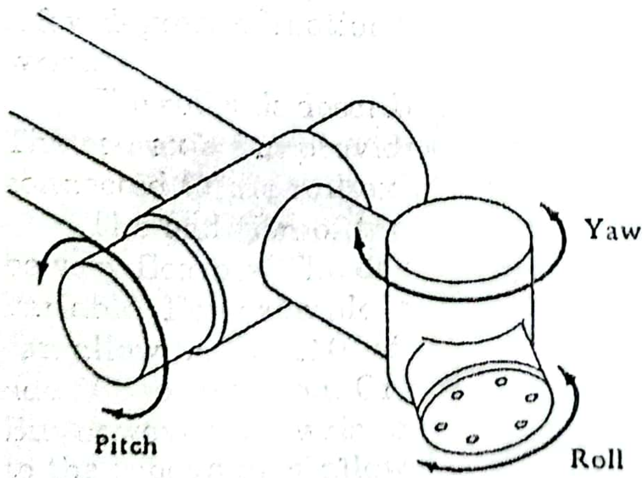


Figure 2–6 Wrist Axis Motions

Gripper designs must meet the demands of the various tasks that the grippers must perform, such as holding different-shaped parts. For instance, the gripper should be designed to hold the part without causing damage to the part. Also, the gripper must have fingers that will hold the part in position as the part is moved from one location to another. Finally, the gripper should be flexible enough to grasp a family of different parts.

Grippers can be designed to grasp a part on the inside diameter of the part or on the outside diameter of the part. The gripper should be able to hold the part securely as the manipulator moves into position. The gripper must make contact with the part in several different places. The gripper must also supply enough gripping force to the part to overcome the effects of gravity on the part.

In the design of the gripper, additional weight must be accounted for because of the gravitational pull of the earth and the acceleration of the manipulator. These two factors may mean that the weight of the part is three times its normal weight. Other variables that must be taken into account by the designer are the center of the part to be gripped, how far the gripping point is from the center of gravity of the part, the size of the pads that will grip the part, and the weight of the part to be lifted by the gripper.

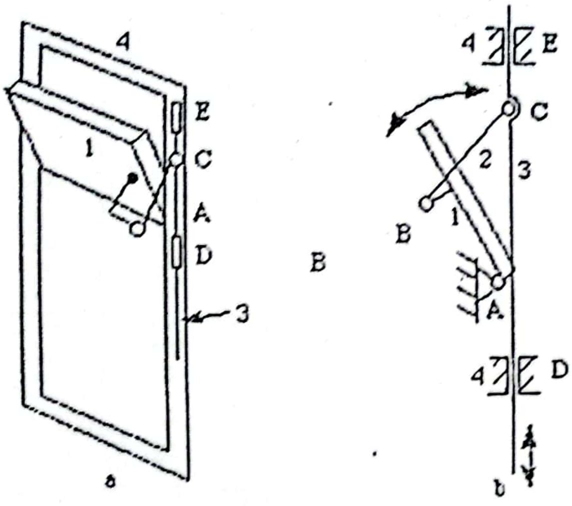
Mechanical grippers, which are the type used most often in industrial applications, are driven by either pneumatic actuators or gearing systems. The mechanical gripper is used to lift various sizes and weights of parts.

In many applications, a mechanical gripper will not provide the proper grasping method for the part. In these cases, a vacuum gripper can be used to hold the part in position while the part is transferred from one location to another. The vacuum gripper has rubber cups that hold the part and a vacuum, or negative-pressure, system. The suction created by the vacuum holds the part in position while the manipulator is in motion.

The magnetic gripper is used to pick up parts made of ferrous metals. When the magnetic field of the gripper comes into contact with the ferrous metal part, it induces a magnetic field of opposite polarity into the part. Thus, the part and the gripper attract each other, allowing the part to be lifted.

End-of-arm tooling is a tool connected to the end effector flange of the manipulator. These tools allow the manipulator to perform tasks such as arc welding, spot welding, drilling, routing, deburring, and sealing.

… ansom above the door in Figure a. The opening and closing mechanism is shown in Figure b. Let's calculate its degree of freedom.



Transom mechanism

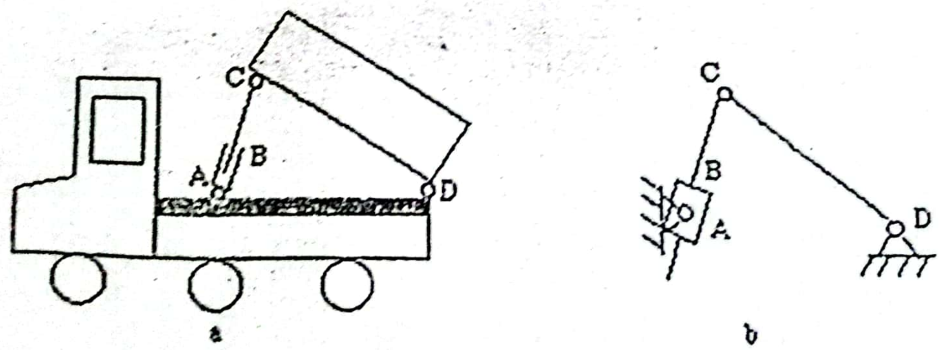
$n = 4$ (link 1,3,3 and frame 4), $l = 4$ (at A, B, C, D), $h = 0$

$$F = 3(4-1) - 2 \times 4 - 1 \times 0 = 1$$

Note: D and E function as a same prismatic pair, so they only count as one lower pair.

Example 2

Calculate the degrees of freedom of the mechanisms shown in Figure b. Figure a is an application of the mechanism.



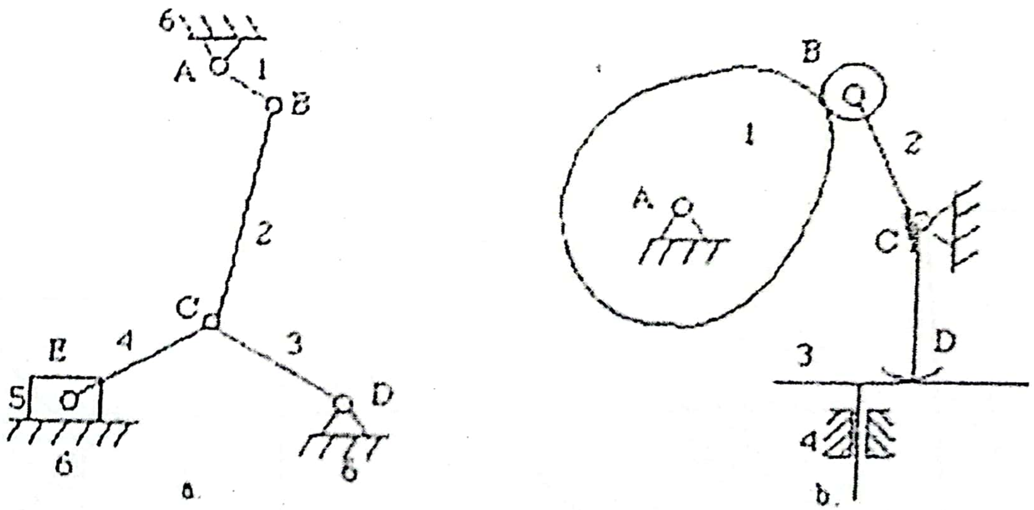
Dump truck

$n = 4, l = 4$ (at A, B, C, D), $h = 0$

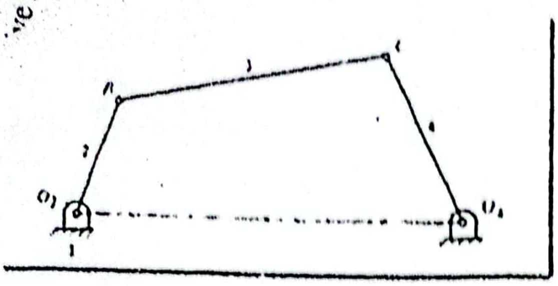
$$\therefore F = 3(4-1) - 2\times 4 - 1\times 0 = 1$$

Example 3

Calculate the degrees of freedom of the mechanisms shown in Figure



Degrees of freedom calculation



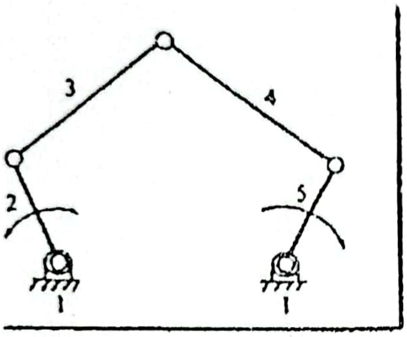
$F = 3(n-1)-2l-h$

Here, $n_2 = 4$, $n = 4$, $l = 4$ and $h = 0$.

$F = 3(4-1)-2(4) = 1$

I.e., one input to any one link will result in definite motion of all the links.

(ii)



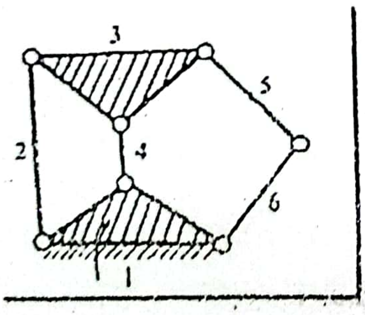
$F = 3(n-1)-2l-h$

Here, $n_2 = 5$, $n = 5$, $l = 5$ and $h = 0$.

$F = 3(5-1)-2(5) = 2$

I.e., two inputs to any two links are required to yield definite motions in all the links.

(iii)



$F = 3(n-1)-2l-h$

Here, $n_2 = 4$, $n_3 = 2$, $n = 6$, $l = 7$ and $h = 0$.

$F = 3(6-1)-2(7) = 1$

I.e., one input to any one link will result in definite motion of all the links.

(iv)

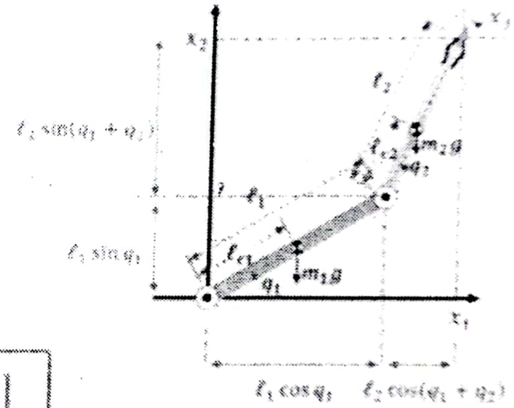
ıple: Forward kinematics

coordinates of the end-effector:

$$x_1 = \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2)$$
$$x_2 = \ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2)$$

orientation of the end effector:

$$x_3 = q_1 + q_2$$



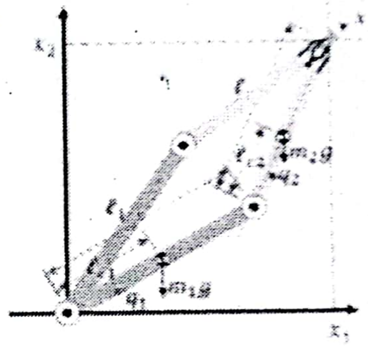
$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \varphi(\boldsymbol{q}) = \begin{bmatrix} \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2) \\ \ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2) \\ q_1 + q_2 \end{bmatrix}$$

ıple: Inverse kinematics

l effector at (x_1, x_2)

$$q_1 = \tan^{-1}\left(\frac{y_e}{x_e}\right) \mp \tan^{-1}\left(\frac{\ell_2 \sin q_2}{\ell_1 + \ell_2 \cos q_2}\right)$$

$$q_2 = \pm\cos^{-1}\left(\frac{x_e^2 + y_e^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2}\right)$$



te that there are only two possible values for x_3.

Necessary Storage tank in pneumatic actuator

A compressed air tank or air receiver, the primary function of an air reservoir is to act as temporary storage to accommodate peak demands of compressed air. By being able to to handle any sudden or unusually heavy demands in excess capacity r, air reservoirs can act as a buffer between your system and any fluctuation in pressure. Air reservoirs can also serve to dampen pulsations from discharge lines of your compressed air system, resulting in steadier pressure. They can also help prevent frequent loading and unloading of compressed air systems, as well as precipitating any moisture or oil carryover within any compressed air generated.

Compressors have a tank connected to store the air before it's released into the pneumatic track. Buffer tanks are secondary storage units for the compressed air that came from the compressor. They are storing the high-PSI (pounds per square inch) compressed air for the pneumatic actuators. These tanks help in preventing irregular airflow surges in the actuators, allowing the compressor cycle to maximise its shutoff timing. They also allow the compressor to be in the exact distance from the actuators in projects.

Trajectory planning

Trajectory planning consists in finding a time series of successive joint angles that allows moving a robot from a starting configuration towards a goal configuration, in order to achieve a task, such as grabbing an object from a conveyor belt and placing it on a shelf.

Acceleration analysis in robotics

Acceleration: (Robotic Mechanisms) The time rate of change of velocity of a body. It is always produced by force acting on a body. Acceleration is measured as feet per second per second (ft/s2) or meters per second per second (m/s2). These three concepts are important for understanding kinematics and the variables and formulas used to solve different kinematic problems

Laplace transform is important in control system

The Laplace transform is extensively used in control theory. It appears in the description of linear time-invariant systems, where it changes convolution operators into multiplication operators and allows one to de.ne the transfer function of a system.

Laplace transform is a mathematical tool that can simplify the analysis and design of control systems. It can convert complex differential equations that describe the dynamic behavior of a system into simpler algebraic equations that describe the frequency response of a system

Mathematical modeling of a physical system

Mathematical modeling and representation of a physical system. A physical system is a system in which physical objects are connected to perform an objective. We cannot represent any physical system in its real form. Therefore, we have to make assumptions for analysis and synthesis of systems.

A mathematical model is the mathematical representation of the physical system which is made using the appropriate governing laws of that system. These governing laws are Newton's laws of motion

the effects of integral and derivative control actions on system performance

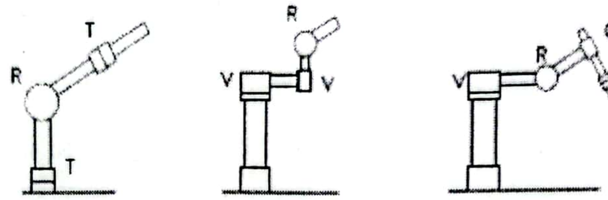
An integral control (����) will have the effect of eliminating the steady-state error for a constant or step input, but it may make the transient response slower. A derivative control (����) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response

PROBLEMS

Robot Anatomy

8.1 Using the notation scheme for defining manipulator configurations (Section 8.1.2), draw diagrams (similar to Figure 8.1) of the following robots: (a) TRT, (b) VVR, (c) VROT.

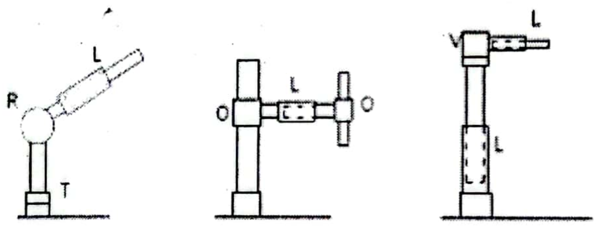
Solution



(a) TRT (b) VVR (c) VROT

8.2 Using the notation scheme for defining manipulator configurations (Section 8.1.2), draw diagrams (similar to Figure 8.1) of the following robots: (a) TRL, (b) OLO, (c) LVL.

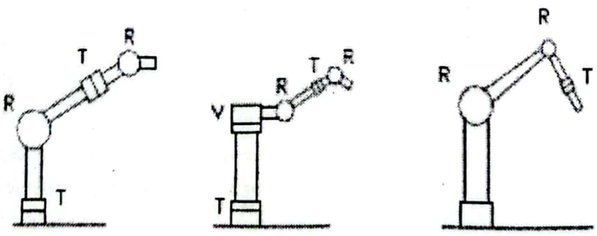
Solution



(a) TRL (b) OLO (c) LVL

8.3* Using the notation scheme for defining manipulator configurations (Section 8.1.2), draw diagrams (similar to Figure 8.1) of the following robots: (a) TRT:R, (b) TVR:TR, (c) RR:T.

Solution



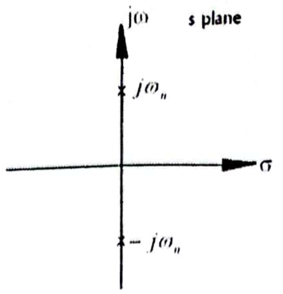
(a) TRT:R (b) TVR:TR (c) RR:T

8.4 Using the robot configuration notation scheme discussed in Section 8.1, write the configuration notations for some of the robots in your laboratory or shop.

Solution: Answer depends on robots in the laboratory or shop of interest.

8.5 Describe the differences in orientation capabilities and work volumes for a TR and a RT wrist assembly. Use sketches as needed.

Solution



Solved problems:

1. A single degree of freedom spring-mass-damper system has the following data: spring stiffness 20 kN/m; mass 0.05 kg; damping coefficient 20 N-s/m. Determine
   (a) undamped natural frequency in rad/s and Hz
   (b) damping factor
   (c) damped natural frequency n rad/s and Hz.

If the above system is given an initial displacement of 0.1 m, trace the phasor of the system for three cycles of free vibration.

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{20\times10^3}{0.05}} = 632.46 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{632.46}{2\pi} = 100.66 \text{ Hz}$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{20\times10^3\times0.05}} = 0.32$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 632.46\sqrt{1-0.32^2} = 600 \text{ rad/s}$$

$$f_d = \frac{\omega_d}{2\pi} = \frac{600}{2\pi} = 95.37 \text{ Hz}$$

$$y(t) = Ae^{-\zeta\omega_n t} = 0.1e^{-0.32\times632.46t}$$

2. A second-order system has a damping factor of 0.3 (underdamped system) and an un-damped natural frequency of 10 rad/s. Keeping the damping factor the same, if the un-damped natural frequency is changed to 20 rad/s, locate the new poles of the system? What can you say about the response of the new system?

Solution:

Given, $\omega_{n1} = 10$ rad/s and $\omega_{n2} = 20$ rad/s

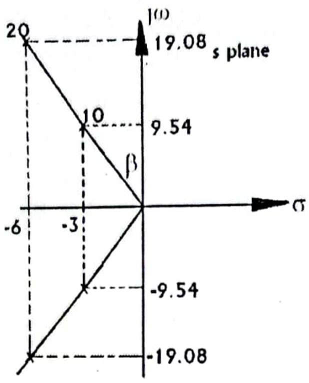
$$\omega_{d_1} = \omega_{n_1}\sqrt{1-\zeta^2} = 10\sqrt{1-0.3^2} = 9.54 \text{ rad/s}$$

$$\omega_{d_2} = \omega_{n_2}\sqrt{1-\zeta^2} = 20\sqrt{1-0.3^2} = 19.08 \text{ rad/s}$$

$$p_{1,2} = -\zeta\omega_{n_1} \pm j\omega_{d_1} = -3 \pm j9.54$$

$$p_{3,4} = -\zeta\omega_{n_2} \pm j\omega_{d_2} = -6 \pm j19.08$$

$$\tan\beta = \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{0.3}{\sqrt{1-0.3^2}} = 17.45^o$$



8.9.1. Second-order Time Response Specifications with Impulse input

(a) Over damped case (ζ>1)

General equation

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{Kx_i}{m}\delta(t) \quad (8.76)$$

Laplacian of the output

$$Y(s) = \frac{Kx_i}{m}\left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

$$= \frac{Kx_i}{2m\omega_n\sqrt{\zeta^2-1}}\left\{\frac{1}{(s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})} - \frac{1}{(s+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}}\right\} \quad (8.77)$$

Time-domain response

$$y(t) = \left[\frac{Kx_i}{m\omega_n\sqrt{\zeta^2-1}}\right]e^{-\zeta\omega_n t}\sinh\left(\omega_n\sqrt{\zeta^2-1}\right)t \quad (8.78)$$

(b) Critically damped case (ζ=1)

General equation

$$\ddot{y} + \omega_n^2 y = \frac{Kx_i}{m}\delta(t) \quad (8.79)$$

Laplacian of the output

$$Y(s) = \frac{Kx_i}{m}\left(\frac{1}{s^2 + \omega_n^2}\right) \quad (8.80)$$

Time-domain response

$$y(t) = \left\{\frac{Kx_i}{m\omega_n}\right\}\omega_n t e^{-\omega_n t} \quad (8.81)$$

(c) Under damped case (ζ<1)

Table 2.3 Comparison of fundamental robot arms [Courtesy: Fuller (1999)]

Configuration	Advantages	Disadvantages
Cartesian (3 linear axes) x: base travel y: height z: reach	- Easy to visualize - Rigid structure - Easy offline programming - Easy mechanical stops	- Reach only front and back - Requires large floor space - Axes are hard to seal - Expensive
Cylindrical (1 rotation and 2 linear axes) θ : base rotation y: height z : reach	- Can reach all around - Rigid y, z-axes - θ-axis easy to seal	- Cannot reach above itself - Less rigid θ-axis - y, z-axes hard to seal - Won't reach around obstacles - Horizontal motion is circular
Spherical (2 rotating and 1 linear axes) θ: base rotation ϕ: elevation angle z: reach	- Can reach all around - Can reach above or below obstacles - Large work volume	- Cannot reach above itself - Short vertical reach
Articulated (3 rotating axes) θ: base rotation ϕ: elevation angle ψ: reach angle	- Can reach above or below objects - Largest work volume for least floor space	- Difficult to program off-line - Two or more ways to reach a point - Most complex robot

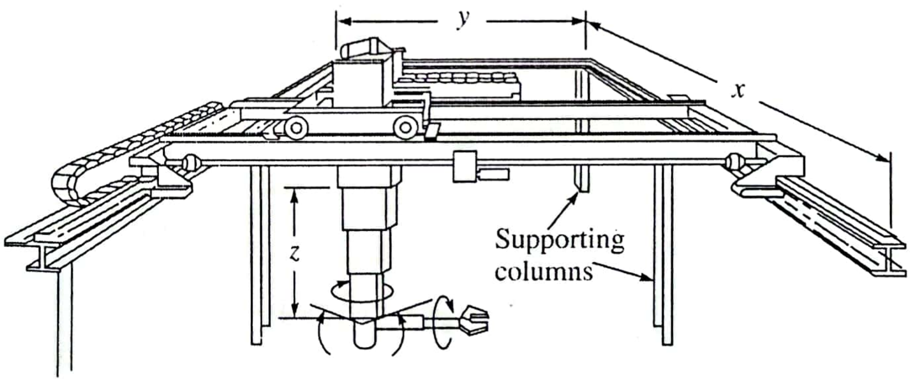


Fig. 2.15 A Gantry robot [Courtesy: Koivo (1989)]

2.2.2 Actuation Systems

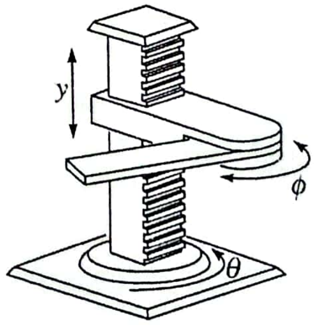
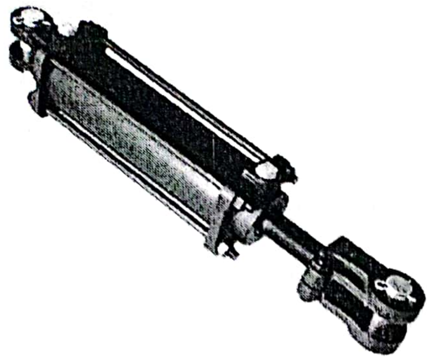


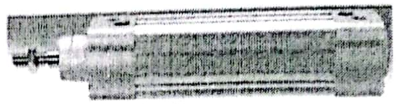
Fig. 2.16 A SCARA arm [Courtesy: Fuller (1999)]

Robots are driven by either electric power or fluid power. The latter category can be further subdivided into pneumatic and hydraulic. Today, the most common drive method is electric with various types of motors, e.g., stepper, dc servo, and brushless ac servo. Pneumatic robots are used in light assembly or packing work but are not usually suitable for heavy-duty tasks or where speed control is necessary. On the other hand, hydraulic



[Courtesy: www.meritindustriesltd.com]

(a) Hydraulic cylinder



[Courtesy: www.festo.com]

(b) Pneumatic cylinder

Fig. 3.25 Commercially available cylinders

3.3 PNEUMATIC ACTUATORS

Pneumatic actuators are the other type of fluid power devices for industrial robots. Pneumatic actuators utilize compressed air for the actuation of cylinders as shown in Fig. 3.25 (b), and are widely used for typical opening and closing motions of jaws in the gripper of a robot, as shown in Fig. 2.4 (a), or for the actuation of simple robot arms used in applications where continuous motion control is not of concern. A pneumatic actuator comprising of a pneumatic cylinder and other accessories are shown in Fig. 3.26, whereas the specifications of a cylinder are given in Table 3.9. Typical advantages and disadvantages of pneumatic actuators are as follows:

Advantages

- It is the cheapest form of all actuators. Components are readily available and compressed air is normally an already existing facility in factories.
- Compressed air can be stored and conveyed easily over long distances.
- Compressed air is clean, explosion-proof and insensitive to temperature fluctuations, thus, lending itself to many applications.
- They have few moving parts making them inherently reliable and reducing maintenance costs.
- Since pneumatic systems are common throughout industry, therefore, relevant personnel are often very familiar with the technology.
- Very quick in action and response time, thus, allowing fast work cycles.
- No mechanical transmission is usually required.
- Pneumatics can be intrinsically safe in explosive areas as no electrical control is required. Also in wet conditions there is no danger of electrocution.
- The systems are usually compact.
- Control is simple, e.g., mechanical stops are often used.
- Individual components can be easily interconnected.

Disadvantages

- Since air is compressible, precise control of speed and position is not easily obtainable unless more complex electromechanical devices are incorporated into the system. This means that only a limited sequence of operation at a fixed speed is often available.

at power levels under about 1.5 kW unless there is danger due to possible ignition of explosive materials. At ranges between 1-5 kW, the availability of a robot in a particular coordinate system with specific characteristics or at a lower cost may determine the decision. Reliability of all types of robots made by reputable manufacturers is sufficiently good that this is not a major determining factor.

Example 3.5 **Selection of a Motor**

Simple mathematical calculations are needed to determine the torque, velocity, and power characteristics of an actuator or a motor for different applications. Torque is defined in terms of force times distance or moment. A force, f, at distance, a, from the center of rotation has a moment or torque, τ, i.e., $\tau = fa$. In general terms, power, P, transmitted in a drive shaft is determined by the torque, τ, multiplied by the angular velocity, ω. Power P is expressed as, $P = \tau\,\omega$. For an example, a calculation can tell one what kilowatt or horsepower is required in a motor used to drive a 2-meter robot arm lifting a 25 kg mass at 10 rpm. If the mass of the arm is assumed zero then, $P = (25 \times 9.81 \times 2) \times (2\pi \times 10/60) = 0.513$ kW. The use of simple equations of this type is often sufficient to make a useful approximation of a needed value. More detailed calculations can take place using the equations of statics and dynamics that apply.

3.5 GRIPPERS

Grippers are end-effectors, as introduced in Section 2.1.1, which are used to grasp an object or a tool, e.g., a grinder, and hold it. Tasks required by the grippers are to hold workpieces and load/unload from/to a machine or conveyer. Grippers can be mechanical in nature using a combination of mechanisms driven by electric, hydraulic, or pneumatic powers, as explained in earlier sections. Grippers can be classified based on the principle of grasping mechanism. For example, grippers can hold with the help of suction cups, magnets, or by other means. A gripper is then accordingly referred to as *pneumatic gripper*, *magnetic gripper*, etc. Another way to classify a gripper is based on how it holds an object, i.e., based on grasping the object on its exterior (*external gripper*) or interior (*internal gripper*) surface.

3.5.1 Mechanical Grippers

As shown in Fig. 2.4(a), mechanical grippers have their jaw movements through pivoting or translational motion using a transmission element, e.g., linkages or gears, etc. This is illustrated in Fig. 3.27. The gripper can be of single or double type. While the former has only one gripping device at the robot's wrist, the latter type has two. The double grippers can be actuated independently and are especially useful in machine loading and unloading. As illustrated in Groover et al. (2012), suppose a particular job calls for a raw part to be loaded from a conveyor onto a machine and the finished part to be unloaded onto another conveyor. With a single gripper, the robot would have to unload the finished part before picking up the raw part. This would consume valuable time in the production cycle because the machine would remain idle during these handling motions. With a double gripper, the robot can pick up the part from the incoming conveyor with one of its gripping devices and have it ready to exchange for the finished part on the machine. When the machine cycle is completed, the robot can reach in for the finished part with the available grasping device, and insert the raw part into the machine with the other grasping device. The amount of time spent in exchanging the parts or the time to keep the machine idle is minimized.

A gripper uses its fingers or jaws to hold an object, as illustrated in Fig. 3.28. The function of a gripper mechanism is to translate some form of power input, be it electric, hydraulic or pneumatic, into the grasping action of the fingers against the part. Note that there are two ways a gripper can hold an object, i.e., either by physical constriction as shown in Fig. 3.28(a) or by friction, as demonstrated in Fig. 3.38(b). In the former case, contacting surfaces of the fingers are made of approximately the same shape of the part geometry, while in the latter case the fingers must apply sufficient force to retain the part against gravity or accelerations. The friction method of holding a part is less complex and hence less expensive. However, they have to be designed properly with its surfaces having sufficient coefficient of friction so that the parts do not slip during motion. Example 3.6 illustrates the situation.

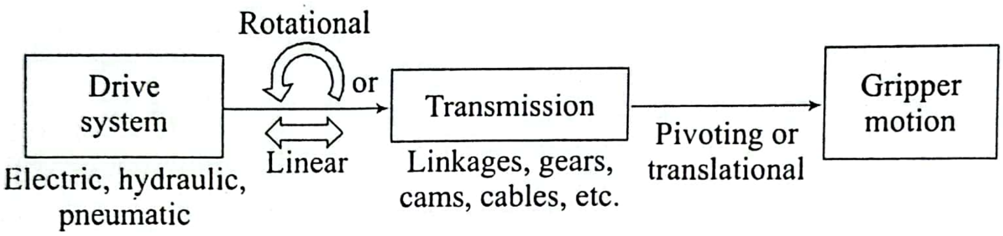


Fig. 3.27 Motion of a mechanical gripper

Example 3.6 Friction-based Gripper

Consider the weight w of an object to be carried by a parallel-fingered gripper shown in Fig. 3.28(b). Gripper force can be calculated using the following force balance:

$$\mu n f = w \qquad (3.5a)$$

where μ is the coefficient of friction of the object and finger surfaces, whereas n, f, and w are the number of contacting surfaces, finger forces, and the weight of the object to be held by the gripper. If w = 140 kg, n = 2, and μ = 0.2, f is computed simply as

$$f = \frac{140 \times 9.81}{2 \times 0.2} \cong 3500 \text{ N} \qquad (3.5b)$$

Note that the value of 9.81 m/s^2 is the value of g, i.e., acceleration due to gravity.

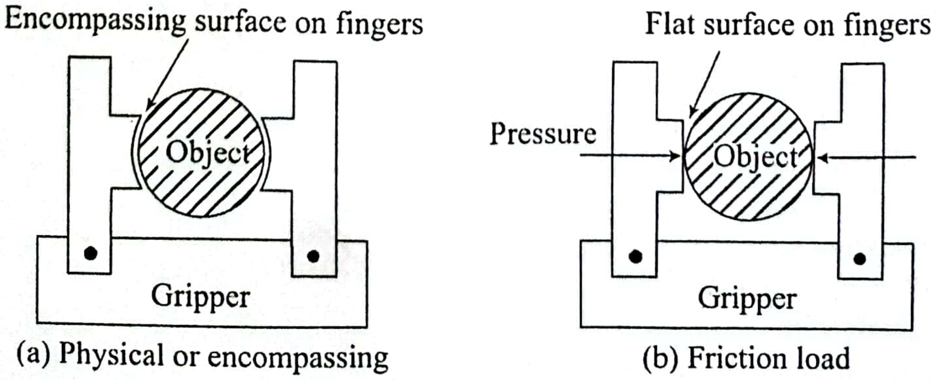
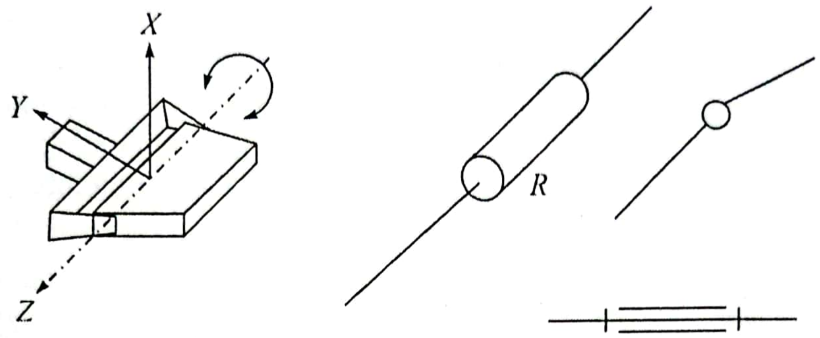


Fig. 3.28 Fingers gripping objects

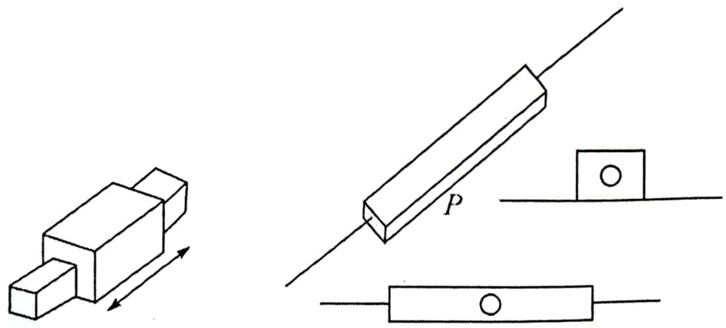
1. Revolute Joint, R A revolute joint, also known as a *turning pair* or a *hinge* or a *pin joint*, permits two paired links to rotate with respect to each other about the axis of the joint, say, Z, as shown in Fig. 5.1. Hence, a revolute joint imposes five constraints, i.e., it prohibits one of the links to translate with respect to the other one along the three perpendicular axes, X, Y, Z, along with the rotation about two axes, X and Y. This joint has one degree of freedom (DOF).



(a) Geometrical form (b) Representations

Fig. 5.1 A revolute joint

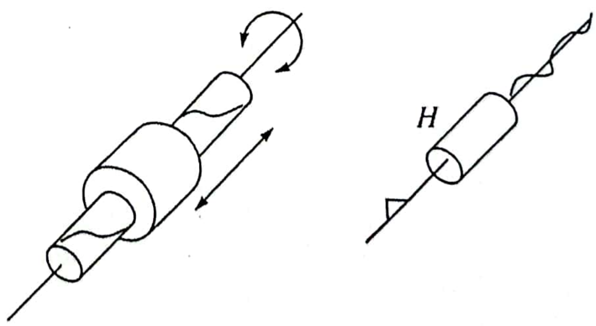
2. Prismatic Joint, P A *prismatic joint* or a *sliding pair* allows two paired links to slide with respect to each other along its axis, as shown in Fig. 5.2. It also imposes five constraints and, hence, has one DOF.



(a) Geometrical form (b) Representations

Fig. 5.2 A prismatic joint

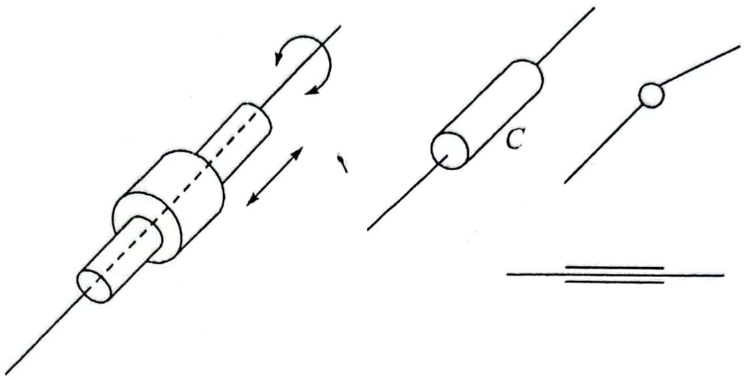
3. Helical Joint, H As shown in Fig. 5.3, a helical joint allows two paired links to rotate about and translate at the same time along the axis of the joint. The translation is, however, not independent. It is related to the rotation by the pitch of the screw. Thus, the helical joint also has five constraints, and accordingly one DOF.



(a) Geometrical form (b) Representations

Fig. 5.3 A helical joint

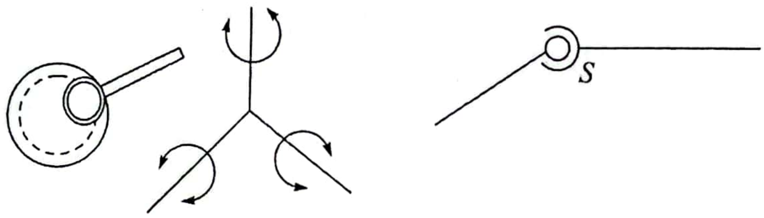
4. Cylindrical Joint, C It permits rotation about, and independent translation along, the axis of the joint, as shown in Fig. 5.4. Hence, a cylindrical joint imposes four constraints on the paired links, and has two DOF.



(a) Geometrical form (b) Representations

Fig. 5.4 A cylindrical joint

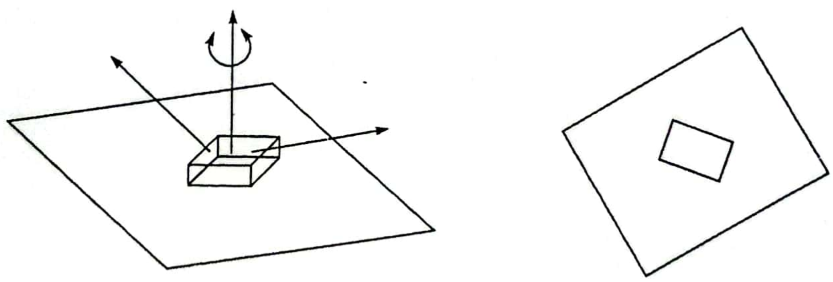
5. Spherical Joint, S It allows one of the coupled links to rotate freely in all possible orientation with respect to the other one about the center of a sphere. No relative translation is permitted. Hence, it imposes three constraints and has three DOF.



(a) Geometrical form (b) Representation

Fig. 5.5 A spherical joint

6. Planar Joint, L This three DOF joint allows two translations along the two independent axes of the plane of contact and one rotation about the axis normal to the plane, Fig. 5.6.



(a) Geometrical form (b) Representation

Fig. 5.6 A planar joint

Table 5.1 Lower pair joints

Name	Symbol	Geometric Form and Representations	DOF	Common Surface
Revolute	*R*	Fig. 5.1	1	Cylinder
Prismatic	*P*	Fig. 5.2	1	Prism
Helical	*H*	Fig. 5.3	1	Screw
Cylindrical	*C*	Fig. 5.4	2	Cylinder
Spherical	*S*	Fig. 5.5	3	Sphere
Planar	*L*	Fig. 5.6	3	Plane

Table 5.1 summarizes the basic lower pair joints, where all the joints have surface contact between the interconnecting links. Another commonly used lower pair joint in robotics is the two DOF *universal joint*, as shown in Fig. 5.7. This is the combination of two intersecting revolute joints. Examples of higher pair joints in robots are *gears* and *cams with roller followers*, where they make line contacts.

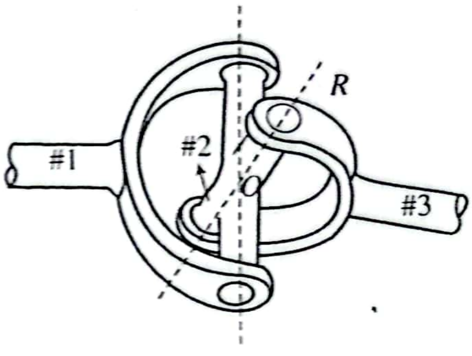


Fig. 5.7 A universal joint

5.1.2 Kinematic Chain

A *kinematic chain* is a series of links connected by joints. When each and every link in a kinematic chain is coupled to at most two other links, the chain is referred to as *simple kinematic chain*. A simple kinematic chain can be either *closed* or *open*. It is closed if each and every link is coupled to two other links as shown in Fig. 5.8. A kinematic chain is open if it contains exactly two links, namely, the end ones that are coupled to only one link. A robotic manipulator shown in Fig. 5.9 falls in this category.

Mechanism vs. Manipulator

Whereas a mechanism can be open- and closed-chain mechanical system; a manipulator is treated here as an open-chain system.

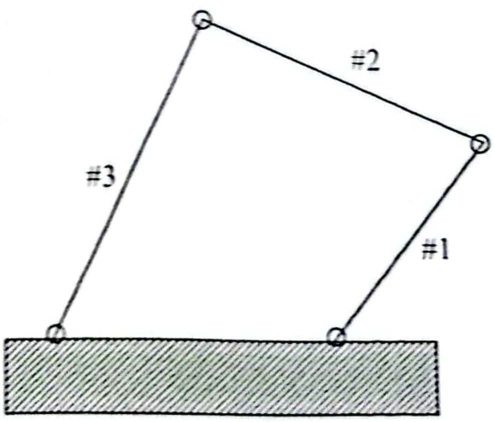


Fig. 5.8 A four-bar mechanism

11.11 FORCE CONTROL

In the control schemes presented so far, the controlling joint torques were obtained using only joint or Cartesian trajectories without any reference to what force or moment the robot exert on its environment. This is important as the robot's end-effector has to exert some moment or force or both, which were defined in Chapter 9 as wrench, on an object handled by it. In many tasks, e.g., cleaning a glass, the robot should not exert more than certain force in order not to break it. In other instances, where robots and human beings are working together, a robot must stop when it is touched, i.e., a threshold value of force is crossed to make the environment safe. In applications where the end-effector has to apply a controlled force, e.g., on a glass which is cleaned by a robot, the position control schemes presented in the previous sections will not be appropriate unless the tool at the end-effector is sufficiently compliant, e.g., like a sponge. If the end-effector is rigid like a scrapping tool then any uncertainty in the position of the glass surface or the positional error in the end-effector will either cause the glass to break or the scrapper will not touch the glass at all. In such situations, the force, not the position, should be specified in order to obtain satisfactory performance by the robot. In fact, the robot motion can be subdivided into *gross* and *fine* motions. While the former is usually fast and generally should be used for position control, the latter is slow when the end-effector's position relative to the environment is accurately adjusted based on force, which could be either *implicit* or *explicit*, based on an external force/torque sensor fitted at the wrist before the end-effector. Since the force control is generally slow, nonlinear dynamics of the robot is of little importance.

Wrench Control

Wrench means both moment and force, as defined in Chapter 9. Force control, which is more popular in the literature, is equivalent to wrench control.

An approach to maintain an interaction force between a robot and its environment is by introducing some kind of compliance into the robot. If this compliance is large, a small position error will cause a small change in interaction force. A special passive compliance device, e.g., Remote Center Compliance (WR: RCC), can be placed in the robot's wrist to control the applied forces onto the environment. Such devices help in automated assembly, e.g., peg-in-hole or grinding tasks. Passive compliance devices can be undesirable as it may lead to oscillations if the manipulator moves very fast. Alternatively, it is possible to introduce compliance by reducing position feedback gain, which is referred as active compliance. In this case, a small manipulator displacement will cause a corresponding small variation in the control torques. Unlike mechanical compliance, as in the case of RCC, such artificial compliance is software-adjustable depending on the specific task. Controlling end-effector wrenches through the joint-actuator torques is known as *implicit force control.* Here, position error is related to the contact force through a mechanical stiffness or impedance with adjustable parameters, i.e., mass, damping ratio, and stiffness of the system. A robot under impedance control is equivalent to an equivalent mass-spring-damper system with contact force as input. One can, however, use an external force/torque sensor to measure its output to feedback in the drive control system. Such control can be referred to as explicit force control.

where $\tilde{\mathbf{I}} \equiv \mathbf{I} - \hat{\mathbf{I}}$ and $\tilde{\boldsymbol{\varphi}} \equiv \boldsymbol{\varphi} - \hat{\boldsymbol{\varphi}}$. Subtracting Eq. (11.32) from the linear form of the dynamic equations given by Eq. (8.67), i.e., $\mathbf{I}\ddot{\boldsymbol{\theta}} + \boldsymbol{\varphi} = \mathbf{Y}\hat{\mathbf{p}}'$, one can also write the following:

$$\tilde{\mathbf{I}}\ddot{\boldsymbol{\theta}} + \tilde{\boldsymbol{\varphi}} = \mathbf{Y}\tilde{\mathbf{p}}, \text{ where } \tilde{\mathbf{p}} \equiv \hat{\mathbf{p}}' - \hat{\mathbf{p}} \tag{11.33c}$$

Assuming $\hat{\mathbf{I}}$ is invertible, the error dynamics can then be expressed as

$$\ddot{\mathbf{e}} + \mathbf{K}_v\dot{\mathbf{e}} + \mathbf{K}_p\mathbf{e} = \boldsymbol{\Gamma}\tilde{\mathbf{p}}, \text{ where } \boldsymbol{\Gamma} \equiv \hat{\mathbf{I}}^{-1}\mathbf{Y} \tag{11.33d}$$

Equation (11.33d) is now a linear system which can be expressed in state-space form, i.e., in the form of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where the state matrix $\mathbf{A}$ and state vector $\mathbf{x}$ have the same structures as in Eq. (11.21c), whereas the matrix $\mathbf{B}$ and vector $\mathbf{u}$ are given by $\mathbf{B} = [\mathbf{O}\ \mathbf{1}]^T$—$\mathbf{O}$ and $\mathbf{1}$ being the zero and identity matrices of compatible sizes, respectively,—and $\mathbf{u} = \boldsymbol{\Gamma}\tilde{\mathbf{p}}$. Based on the concept of Lyapunov's stability criterion, one can then obtain parametric adaptive law in the following form (Spong and Vidyasagar, 2004):

$$\dot{\tilde{\mathbf{p}}} = -\mathbf{K}_{\tilde{p}}^{-1}\boldsymbol{\Gamma}^T\mathbf{B}^T\mathbf{P}\mathbf{x}$$

where the matrices $\mathbf{P}$ and $\mathbf{K}_{\tilde{p}}$ are symmetric positive definite matrices corresponding to the Lyapunov function, $v = \mathbf{x}^T\mathbf{P}\mathbf{x} + \tilde{\mathbf{p}}^T\mathbf{K}_{\tilde{p}}\tilde{\mathbf{p}}$.

11.10 CARTESIAN CONTROL

In the control schemes presented above, the desired trajectory was assumed to be time histories of joint positions, velocities, and accelerations, and the corresponding effort was joint torques. In a robot, however, it is the robot end-effector which actually performs the work or interacts with its environment. Hence, it is more natural to specify the motion of the robot in terms of the position and orientation of the end-effector, its velocities and accelerations, and/or the corresponding forces and moments acting on it. Note here that the end-effector motion of a robot is achieved by controlling a set of joint motions, as also pointed out in Chapter 5, because that is the way the robots are designed. Hence, the joint motion specifications are used for controlling a robot. In case one wishes to specify the Cartesian motions, along with the forces and moments acting on the end-effector, the controller should have the ability to do the necessary transformation from the Cartesian space of the end-effector to the joint space so that appropriate signals are communicated to the joint actuators. For Cartesian control, the following three approaches can be typically adopted.

11.10.1 Resolved Motion Control

Resolved motion means that the joint motions are combined and resolved into separately controllable end-effector motion along the three Cartesian axes. Such control enables a user to specify the direction, speed, and acceleration, if required, along any arbitrary path of the end-effector. It simplifies the specification of the sequence of motions for the completion of a task because a human operator is usually more adapted to the Cartesian coordinates than the robot's joint coordinates. Based on what is controlled, resolved motion control is accordingly classified as *resolved-rate* or *resolve-acceleration* or *resolved-force* control. Given the position, orientation, and their first two derivatives of the end-effector of a robot, one approach is to use the inverse kinematics for position, velocity, and acceleration to obtain the joint position, velocity, and acceleration, respectively, and use one of the joint control approaches

Thus, the output from an element may be obtained by multiplying the input signal with the transfer function.

Note : From the transfer function of the individual blocks, the equation of motion of system can be formulated.

26.7 Overall Transfer Function

In the previous article, we have discussed the transfer function of a block. A control system actually consists of several such blocks which are connected in series. The overall transfer function of the series is the product of the individual transfer function. Consider a block diagram of any control system represented by the three blocks as shown in Fig. 26.2.

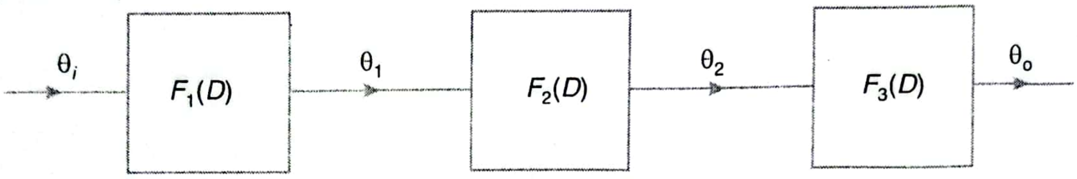


Fig. 26.2. Overall transfer function.

Thus, if $F_1(D)$, $F_2(D)$, $F_3(D)$ are individual transfer functions of three blocks in series, then the overall transfer function of the system is given as

$$\frac{\theta_o}{\theta_i} = \frac{\theta_1}{\theta_i} \times \frac{\theta_2}{\theta_1} \times \frac{\theta_o}{\theta_2} = F_1(D) \times F_2(D) \times F_3(D) = KG(D)$$

where K = Constant representing the overall amplification or gain, and

$G(D)$ = Some function of the operator D.

Note: The above equation is only true if there is no interaction between the blocks, that is the output from one block is not affected by its connection to the subsequent blocks.

26.8. Transfer Function for a System with viscous Damped Output

Consider a shaft, which is used to position a load (which may be pulley or gear) as shown in Fig. 26.3. The movement of the load is resisted by a viscous damping torque.

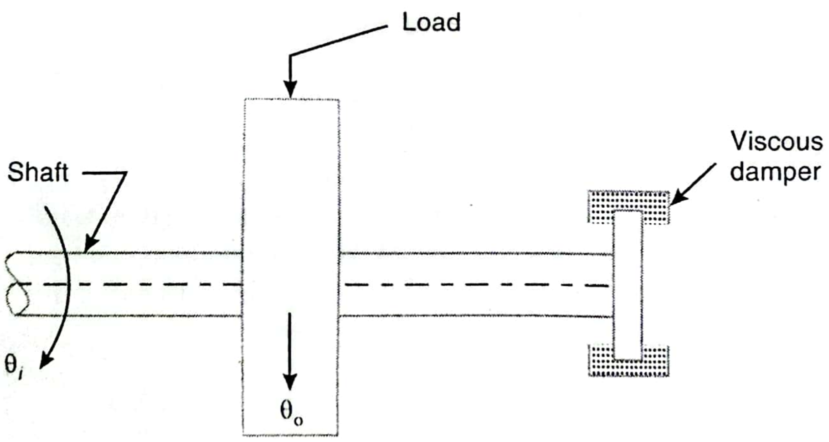


Fig. 26.3. Transfer function for a system with viscous damped output.

Let θ_i = Input signal to the shaft,

θ_o = Output signal of the shaft,

q = Stiffness of the shaft,

I = Moment of Inertia of the load, and

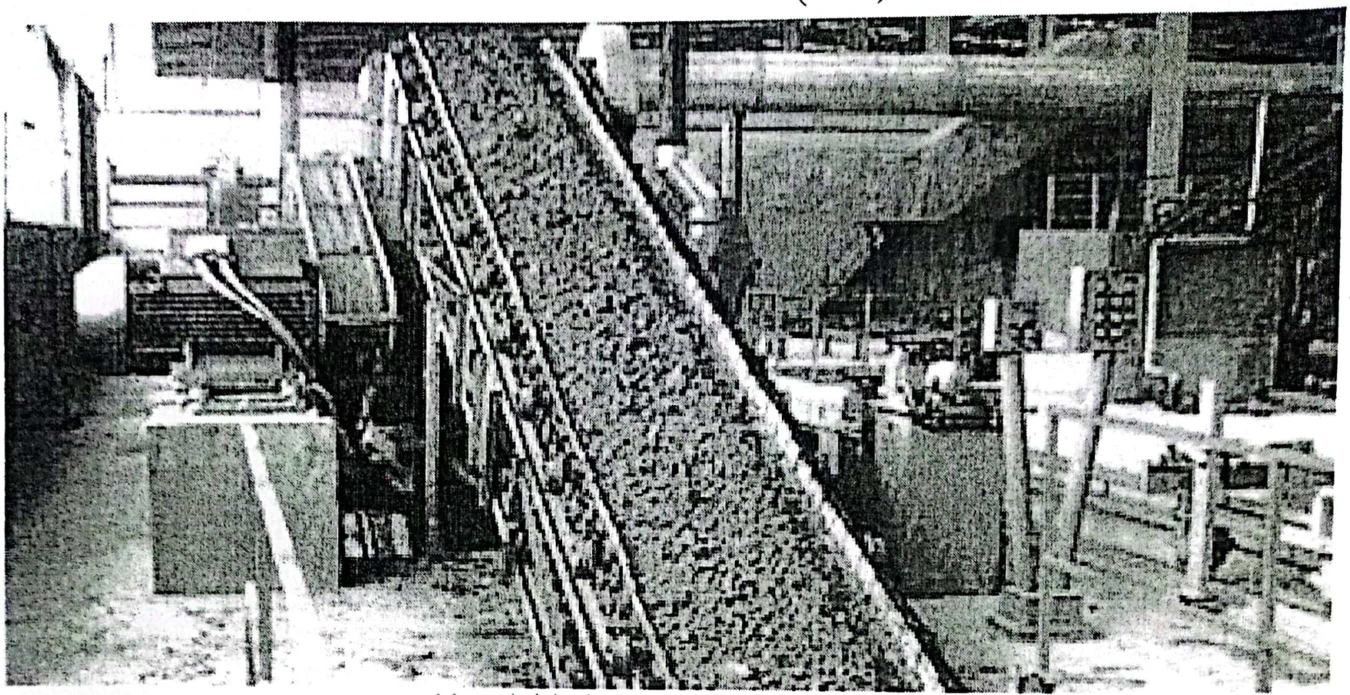
T_d = Viscous damping torque per unit angular velocity.

After some time t,

Twist in the shaft $= \theta_i - \theta_o$

$\therefore$ Torque transmitted to the load $= q(\theta_i - \theta_o)$

We also know that damping torque $= T_d \omega_0 = T_d \left(\frac{d\theta_o}{dt}\right)$ ... $(\because \omega_0 = d\theta_o / dt)$



Material being moved via-belt conveyor.

Note : This picture is given as additional information and is not a direct example of the current chapter.

According to Newton's Second law, the equation of motion of the system is given by

$$I\left(\frac{d^2\theta_o}{dt^2}\right) = q\,(\theta_i - \theta_o) - T_d\left(\frac{d\theta_o}{dt}\right) \quad \text{... } (i)$$

or

$$I\left(\frac{d^2\theta_o}{dt^2}\right) = q\,\theta_i - q\,\theta_o - T_d\left(\frac{d\theta_o}{dt}\right)$$

Replacing d/dt by D in above equation, we get

$$I(D^2\theta_o) = q\,\theta_i - q\,\theta_o - T_d(D\theta_o)$$

or

$$I(D^2\theta_o) + T_d(D\theta_o) + q\,\theta_o = q\,\theta_i$$

$$D^2\theta_o + \frac{T_d}{I}(D\,\theta_o) + \frac{q}{I}(\theta_o) = \frac{q}{I}(\theta_i)$$

$$D^2\theta_o + \frac{T_d}{I}(D\,\theta_o) + (\omega_n)^2\theta_o = (\omega_n)^2\theta \quad \text{... } (ii)$$

where ω_n = Natural frequency of the shaft $= \sqrt{\frac{q}{I}}$

Also we know that viscous damping torque per unit angular velocity,

$$T_d = 2I\xi\omega_n \quad \text{or} \quad T_d / I = 2\xi\omega_n$$

where ξ = Damping factor or damping ratio.

The equation (*ii*) may now be written as

$$D^2\theta+2\xi\omega_n(D\theta_o)+(\omega_n)^2\theta_o=(\omega_n)^2\theta_i$$

or

$$[D^2+2\xi\omega_n D+(\omega_n)^2]\theta_o=(\omega_n)^2\theta_i$$

$\therefore$

$$\text{Transfer function}=\frac{\theta_o}{\theta_i}=\frac{(\omega_n)^2}{D^2+2\xi\omega_n D+(\omega_n)^2}$$

$$=\frac{1}{T^2D^2+2\xi T D+1}$$

where T = Time constant = $1/\omega_n$

Note: The time constant (T) may also be obtained by dividing the periodic time (t_d) of the undamped natural oscillations of the system by 2π. Mathematically,

$$T=\frac{t_d}{2\pi}=\frac{2\pi}{\omega_n}\times\frac{1}{2\pi}=\frac{1}{\omega_n} \qquad \ldots\left(\because\ t_d=\frac{2\pi}{\omega_n}\right)$$

Example 26.1. *The motion of a pointer over a scale is resisted by a viscous damping torque of magnitude 0.6 N-m at an angular velocity of 1 rad/s. The pointer, of negligible inertia, is mounted on the end of a relatively flexible shaft of stiffness 1.2 N-m/rad, and this shaft is driven through a 4 to 1 reduction gear box. Determine its overall transfer function.*

If the input shaft to the gear box is suddenly rotated through 1 completed revolution, determine the time taken by the pointer to reach a position within 1 percent of its final value.

Solution. Given:

T_d = 0.6/1 = 0.6 N-ms/rad;

q = 1.2 N-m/rad

The control system along with its block diagram is shown in Fig 26.4 (*a*) and (*b*) respectively.

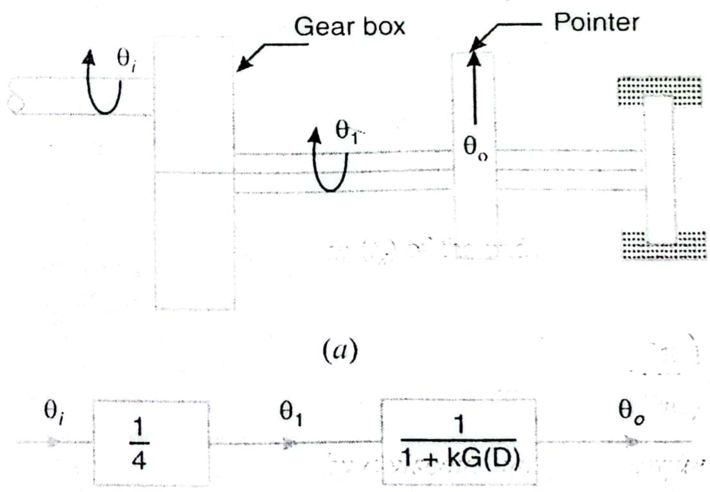


Fig. 26.4

1. *Overall transfer function*

Since the inertia of the pointer is negligible, therefore the torque generated by the twisting of the shaft has only to overcome the damping torque.

Therefore

$$q(\theta_1-\theta_o)=T_d(d\theta_o/dt)$$

where θ_1 = Output from the gear box.

$$\therefore\quad q\theta_1-q\theta_o=T_d(D\theta_o) \qquad \ldots(\because\ d/dt=D)$$

or

$$(q+T_dD)\theta_o=q\theta_1$$

$$\therefore\quad \frac{\theta_o}{\theta_1}=\frac{q}{q+T_dD}=\frac{1}{1+(T_d/q)D}=\frac{1}{1+T\,D} \qquad \ldots(i)$$

where T = Time constant = T_d/q = 0.6/1.2 = 0.5s the behaviour that could be expected of the vehicle when driven over a rough road and hence as a basis for the design of the vehicle suspension. Figure 17.3(c) shows how this model can be used as part of a larger model to predict how the driver might feel when driven along a road. The procedure to be adopted for the analysis of such models is just the same as outlined above for the simple spring–dashpot–mass model. A free-body diagram is drawn for each mass in the system, such diagrams showing each mass independently and just the forces acting on it. Then for each mass the resultant of the forces acting on it is equated to the product of the mass and the acceleration of the mass.

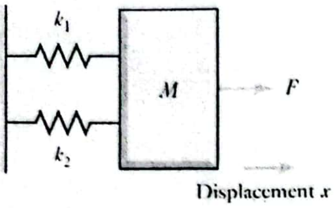


Figure 17.4 Example.

To illustrate the above, consider the derivation of the differential equation describing the relationship between the input of the force F and the output of displacement x for the system shown in Figure 17.4.

The net force applied to the mass is F minus the resisting forces exerted by each of the springs. Since these are k_1x and k_2x, then

$$\text{net force} = F - k_1x - k_2x$$

Since the net force causes the mass to accelerate, then

$$\text{net force} = m\frac{d^2x}{dt^2}$$

Hence

$$m\frac{d^2x}{dt^2} + (k_1 + k_2)x = F$$

The procedure for obtaining the differential equation relating the inputs and outputs for a mechanical system consisting of a number of components can be summarised as:

1 isolate the various components in the system and draw free-body diagrams for each;
2 hence, with the forces identified for a component, write the modelling equation for it;
3 combine the equations for the various system components to obtain the system differential equation.

As an illustration, consider the derivation of the differential equation describing the motion of the mass m_1 in Figure 17.5(a) when a force F is applied. Consider the free-body diagrams (Figure 17.5(b)). For mass m_2 these are the force F and the force exerted by the upper spring. The force exerted by the upper spring is due to its being stretched by $(x_2 - x_3)$ and so is $k_2(x_3 - x_2)$. Thus the net force acting on the mass is

$$\text{net force} = F - k_2(x_3 - x_2)$$

This force will cause the mass to accelerate and so

$$F - k_2(x_3 - x_2) = m_2\frac{d^2x_3}{dt}$$

For the free-body diagram for mass m_1, the force exerted by the upper spring is $k_2(x_3 - x_2)$ and that by the lower spring is $k_1(x_1 - x_2)$. Thus the net force acting on the mass is

$$\text{net force} = k_1(x_2 - x_1) - k_2(x_3 - x_2)$$

time constant is the time taken for a first-order system output to change from 0 to 0.63 of its final steady-state value. In this case this time is about 3 s. We can check this value, and that the system is first order, by finding the value at 2, i.e. 6 s. With a first-order system it should be 0.86 of the steady-state value. In this case it is. The steady-state output is 10 V. Thus the steady-state gain G_{SS} is (steady-state output/input) = 10/5 = 2. The differential equation for a first-order system can be written as

$$\tau \frac{dx}{dt} + x = G_{SS}y$$

Thus, for this system, we have

$$3\frac{dv_o}{dt} + v_o = 2v_i$$

19.4 Second-order systems

Many second-order systems can be considered to be analogous to essentially just a stretched spring with a mass and some means of providing damping. Figure 19.8 shows the basis of such a system.

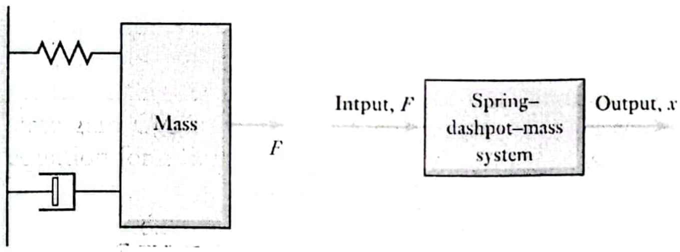


Figure 19.8 Spring–dashpot–mass system.

Such a system was analysed in Section 17.2.2. The equation describing the relationship between the input of force F and the output of a displacement x is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$

where m is the mass, c the damping constant and k the spring constant.

The way in which the resulting displacement x will vary with time will depend on the amount of damping in the system. Thus if the force was applied as a step input and there was no damping at all then the mass would freely oscillate on the spring and the oscillations would continue indefinitely. No damping means $c = 0$ and so the dx/dt term is zero. However, damping will cause the oscillations to die away until a steady displacement of the mass is obtained. If the damping is high enough there will be no oscillations and the displacement of the mass will just slowly increase with time and gradually the mass will move towards its steady displacement position. Figure 19.9 shows the general way that the displacements, for a step input, vary with time with different degrees of damping.

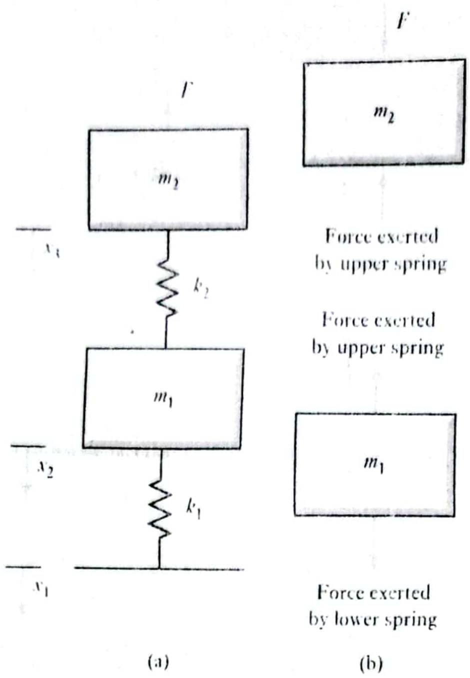


Figure 17.5 Mass–spring system.

This force will cause the mass to accelerate and so

$$k_1(x_2 - x_1) - k_2(x_3 - x_2) = m_1\frac{d^2x_2}{dt}$$

We thus have two simultaneous second-order differential equations to describe the behaviours of the system.

Similar models can be constructed for rotating systems. To evaluate the relationship between the torque and angular displacement for the system the procedure to be adopted is to consider just one rotational mass block, and just the torques acting on that body. When several torques act on a body simultaneously, their single equivalent resultant can be found by addition in which the direction of the torques is taken into account. Thus a system involving a torque being used to rotate a mass on the end of a shaft (Figure 17.6(a)) can be considered to be represented by the rotational building blocks shown in Figure 17.6(b). This is a comparable situation with that analysed above (Figure 17.2) for linear displacements and yields a similar equation

$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = T$$

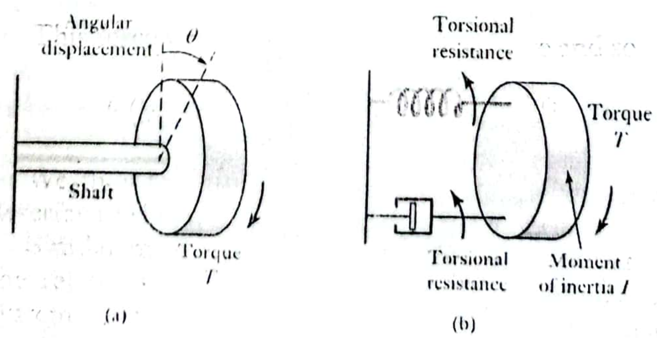


Figure 17.6 Rotating a mass on the end of a shaft: (a) physical situation, (b) building block model.

Now consider when we have damping. The motion of the mass is then described by

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

To solve this equation we can try a solution of the form $x_n = Ae^{st}$. This gives $dx_n/dt = Ase^{st}$ and $d^2x_n/dt^2 = As^2e^{st}$. Thus, substituting these values in the differential equation gives

$$mAs^2e^{st} + cAse^{st} + kAe^{st} = 0$$

$$ms^2 + cs + k = 0$$

Thus $x_n = Ae^{st}$ can only be a solution provided the above equation equals zero. This equation is called the **auxiliary equation.** The roots of the equation can be obtained by factoring or using the formula for the roots of a quadratic equation. Thus

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$= -\frac{c}{2m} \pm \sqrt{\frac{k}{m}\left(\frac{c^2}{4mk}\right) - \frac{k}{m}}$$

But $\omega_n^2 = k/m$ and so, if we let $\zeta^2 = c^2/4mk$, we can write the above equation as

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

ζ is termed the **damping factor.**

The value of s obtained from the above equation depends very much on the value of the square root term. Thus when ζ^2 is greater than 1 the square root term gives a square root of a positive number, and when ζ^2 is less than 1 we have the square root of a negative number. The damping factor determines whether the square root term is a positive or negative number and so the form of the output from the system.

1 *Over-damped*

With $\zeta > 1$ there are two different real roots s_1 and s_2:

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

and so the general solution for x_n is

$$x_n = Ae^{s_1t} + Be^{s_2t}$$

For such conditions the system is said to be **over-damped.**

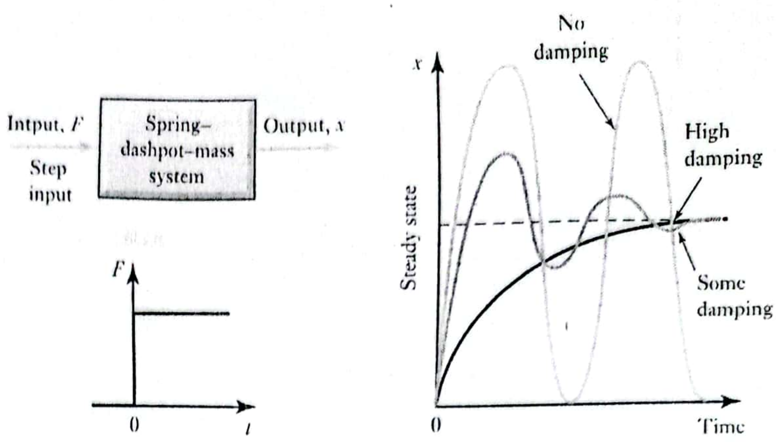
2 *Critically damped*

When $\zeta = 1$ there are two equal roots with $s_1 = s_2 = -\omega_n$. For this condition, which is called **critically damped,**

$$x_n = (At + B)\,e^{-\omega_n t}$$

It may seem that the solution for this case should be $x_n = Ae^{st}$, but two constants are required and so the solution is of this form.

Figure 19.9 Effect of damping with a second-order system.



19.4.1 Natural response

Consider a mass on the end of a spring. In the absence of any damping and left to oscillate freely without being forced, the output of the second-order system is a continuous oscillation (simple harmonic motion). Thus, suppose we describe this oscillation by the equation

$$x = A \sin \omega_n t$$

where x is the displacement at a time t, A the amplitude of the oscillation and ω_n the angular frequency of the free undamped oscillations. Differentiating this gives

$$\frac{dx}{dt} = \omega_n A \cos \omega_n t$$

Differentiating a second time gives

$$\frac{d^2x}{dt^2} = -\omega_n^2 A \sin \omega_n t = -\omega_n^2 x$$

This can be reorganised to give the differential equation

$$\frac{d^2x}{dt^2} + \omega_n^2 x = 0$$

But for a mass m on a spring of stiffness k we have a restoring force of kx and thus

$$m\frac{d^2x}{dt^2} = -kx$$

This can be written as

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Thus, comparing the two differential equations, we must have

$$\omega_n^2 = \frac{k}{m}$$

and $x = A \sin \omega_n t$ is the solution to the differential equation.

then we must have

$$m\frac{d^2x_f}{dt^2}+c\frac{dx_f}{dt}+kx_f=F$$

The previous section gave the solutions for the natural part of the solution. To solve the forcing equation,

$$m\frac{d^2x_f}{dt^2}+c\frac{dx_f}{dt}+kx_f=F$$

we need to consider a particular form of input signal and then try a solution. Thus for a step input of size F at time $t = 0$ we can try a solution $x_f = A$, where A is a constant (see Section 19.3.2 on first-order differential equations for a discussion of the choice of solutions). Then $dx_f/dt = 0$ and $d^2x_f/dt^2 = 0$. Thus, when these are substituted in the differential equation, $0 + 0 + kA = F$ and so $A = F/k$ and $x_f = F/k$. The complete solution, the sum of natural and forced solutions, is thus for the over-damped system

$$x = Ae^{s_1t} + Be^{s_2t} + \frac{F}{k}$$

for the critically damped system

$$x = (At + B)e^{-\omega_n t} + \frac{F}{k}$$

and for the under-damped system

$$x = e^{-\zeta\omega_n t}(P\cos\omega t + Q\sin\omega t) + \frac{F}{k}$$

When $t \to \infty$ the above three equations all lead to the solution $x = F/k$. This is the **steady-state condition.**

Thus a second-order differential equation in the form

$$a_2\frac{d^2x}{dt^2}+a_1\frac{dx}{dt}+a_0x=b_0y$$

has a natural frequency given by

$$\omega_n^2=\frac{a_0}{a_2}$$

and a damping factor given by

$$\zeta^2=\frac{a_1^2}{4a_2a_0}$$

19.4.3 Examples of second-order systems

The following examples illustrate the points made above.

Consider a series RLC circuit (Figure 19.10) with $R = 100\ \Omega$, $L = 2.0$ H and $C = 20\ \mu$F. When there is a step input V, the current i in the circuit is given by (see the text associated with Figure 17.9)

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=\frac{V}{LC}$$

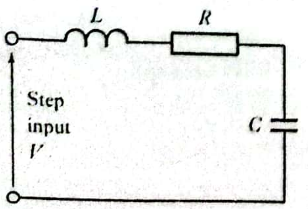


Figure 19.10 RLC system.

3 *Under-damped*
With $\zeta < 1$ there are two complex roots since the roots both involve the square root of (-1):

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_n\sqrt{-1}\sqrt{1 - \zeta^2}$$

and so writing j for $\sqrt{-1}$,

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

If we let

$$\omega = \omega_n\sqrt{1 - \zeta^2}$$

then we can write $s = -\zeta\omega_d \pm j\omega$ and so the two roots are

$$s_1 = -\zeta\omega_d + j\omega \text{ and } s_2 = -\zeta\omega_d - j\omega$$

The term ω is the angular frequency of the motion when it is in the damped condition specified by ζ. The solution under these conditions is thus

$$x_n = Ae^{(\zeta\omega_n + j\omega)t} + Be^{(-\zeta\omega_n - j\omega)t} = e^{-\zeta\omega_n t}(Ae^{j\omega t} + Be^{-j\omega t})$$

But $e^{j\omega t} = \cos\omega t + j\sin\omega t$ and $e^{-j\omega t} = \cos\omega t - j\sin\omega t$. Hence

$$x_n = e^{-\zeta\omega_n t}(A\cos\omega t + jA\sin\omega t + B\cos\omega t - jB\sin\omega t)$$

$$= e^{-\zeta\omega_n t}[(A + B)\cos\omega t + j(A - B)\sin\omega t)]$$

If we substitute constants P and Q for $(A + B)$ and $j(A - B)$, then

$$x_n = e^{-\zeta\omega_n t}(P\cos\omega t + Q\sin\omega t)$$

For such conditions the system is said to be **under-damped**.

19.4.2 Response with a forcing input

When we have a forcing input F the differential equation becomes

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$

We can solve this second-order differential equation by the same method used earlier for the first-order differential equation and consider the solution to be made up of two elements, a transient (natural) response and a forced response, i.e. $x = x_n + x_f$. Substituting for x in the above equation then gives

$$m\frac{d^2(x_n + x_f)}{dt^2} + c\frac{d(x_n + x_f)}{dt} + k(x_n + x_f) = F$$

If we let

$$m\frac{d^2x_n}{dt^2} + c\frac{dx_n}{dt} + kx_n = 0$$

and so

$$\text{subsidence ratio} = \frac{\text{second overshoot}}{\text{first overshoot}} = \exp\left(\frac{-2\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

The settling time t_s is used as a measure of the time taken for the oscillations to die away. It is the time taken for the response to fall within and remain within some specified percentage, e.g. 2%, of the steady-state value (see Figure 19.12). This means that the amplitude of the oscillation should be less than 2% of x_{SS}. We have

$$x = e^{-\zeta\omega_n t}(P\cos\omega t + Q\sin\omega t) + \text{steady-state value}$$

and, as derived earlier, $P = -x_{SS}$. The amplitude of the oscillation is $(x - x_{SS})$ when x is a maximum value. The maximum values occur when ωt is some multiple of π and thus we have $\cos\omega t = 1$ and $\sin\omega t = 0$. For the 2% settling time, the settling time t_s is when the maximum amplitude is 2% of x_{SS}, i.e. $0.02x_{SS}$. Thus

$$0.02x_{SS} = e^{-\zeta\omega_n t_s}(x_{SS} \times 1 + 0)$$

Taking logarithms gives $\ln 0.02 = -\zeta\omega_n t_s$ and since $\ln 0.02 = -3.9$ or approximately -4, then

$$t_s = \frac{4}{\zeta\omega_n}$$

The above is the value of the settling time if the specified percentage is 2%. If the percentage is 5% the equation becomes

$$t_s = \frac{3}{\zeta\omega_n}$$

Since the time taken to complete one cycle, i.e. the periodic time, is $1/f$, where f is the frequency, and since $\omega = 2\pi f$, then the time to complete one cycle is $2\pi/f$. In a settling time of t_s the number of oscillations that occur is

$$\text{number of oscillations} = \frac{\text{settling time}}{\text{periodic time}}$$

and thus for a settling time defined for 2% of the steady-state value,

$$\text{number of oscillations} = \frac{4/\zeta\omega_n}{2\pi/\omega}$$

Since $\omega = \omega_n\sqrt{(1-\zeta^2)}$, then

$$\text{number of oscillations} = \frac{2\omega_n\sqrt{1-\zeta^2}}{\pi\zeta\omega_n} = \frac{2}{\pi}\sqrt{\frac{1}{\zeta^2} - 1}$$

To illustrate the above, consider a second-order system which has a natural angular frequency of 2.0 Hz and a damped frequency of 1.8 Hz. Since $\omega = \omega_n\sqrt{(1-\zeta^2)}$, then the damping factor is given by

$$1.8 = 2.0\sqrt{1-\zeta^2}$$

and $\zeta = 0.44$. Since $\omega t_r = \frac{1}{2}\pi$, then the 100% rise time is given by

$$t_r = \frac{\pi}{2 \times 1.8} = 0.87 \text{ s}$$

If we compare the equation with the general second-order differential equation of

$$a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = b_0y$$

then the natural angular frequency is given by

$$\omega_n^2 = \frac{1}{LC} = \frac{1}{2.0 \times 20 \times 10^{-6}}$$

and so $\omega_n = 158$ Hz. Comparison with the general second-order equation also gives

$$\zeta^2 = \frac{(R/L)^2}{4 \times (1/LC)} = \frac{R^2C}{4L} = \frac{100^2 \times 20 \times 10^{-6}}{4 \times 2.0}$$

Thus $\zeta = 0.16$. Since ζ is less than 1 the system is under-damped. The damped oscillation frequency ω is

$$\omega = \omega_n\sqrt{1 - \zeta^2} = 158\sqrt{1 - 0.16^2} = 156 \text{ Hz}$$

Because the system is under-damped the solution will be of the same form as

$$x = e^{-\zeta\omega_n t}(P\cos\omega t + Q\sin\omega t) + \frac{F}{k}$$

and so

$$i = e^{-0.16\times 158t}(P\cos 156t + Q\sin 156t) + V$$

Since $i = 0$ when $t = 0$, then $0 = 1(P + 0) + V$. Thus $P = -V$. Since $di/dt = 0$ when $t = 0$, then differentiating the above equation and equating it to zero gives

$$\frac{di}{dt} = e^{-\zeta\omega_n t}(\omega P\sin\omega t - \omega Q\cos\omega t) - \zeta\omega_n e^{-\zeta\omega_n t}(P\cos\omega t + Q\cos\omega t)$$

Thus $0 = 1(0 - \omega Q) - \zeta\omega_n(P + 0)$ and so

$$Q = \frac{\zeta\omega_n P}{\omega} = -\frac{\zeta\omega_n V}{\omega} = -\frac{0.16 \times 158V}{156} \approx -0.16 \text{ V}$$

Thus the solution of the differential equation is

$$i = V - Ve^{-25.3t}(\cos 156t + 0.16\sin 156t)$$

Now consider the system shown in Figure 19.11. The input, a torque T, is applied to a disc with a moment of inertia I about the axis of the shaft. The shaft is free to rotate at the disc end but is fixed at its far end. The shaft rotation is opposed by the torsional stiffness of the shaft, an opposing torque of $k\theta_o$ occurring for an input rotation of θ_o. k is a constant. Frictional forces damp the rotation of the shaft and provide an opposing torque of $c\,d\theta_o/dt$, where c is a constant. Suppose we need to determine the condition for this system to be critically damped.

We first need to obtain the differential equation for the system. The net torque is

$$\text{net torque} = T - c\frac{d\theta_o}{dt} - k\theta_o$$

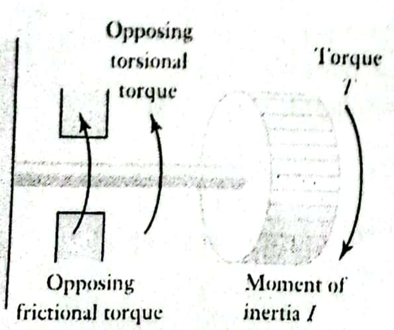


Figure 19.11 Torsional system.

20.2.2 Examples of first-order systems

The following examples illustrate the above points in the consideration of the transfer function of a first-order system and its behaviour when subject to a step input.

1 Consider a circuit which has a resistance R in series with a capacitance C. The input to the circuit is v and the output is the potential difference v_C across the capacitor. The differential equation relating the input and output is

$$v = RC\frac{dv_C}{dt} + v_C$$

Determine the transfer function.

Taking the Laplace transform, with all initial conditions zero, then

$$V(s) = RCsV_C(s) + V_C(s)$$

Hence the transfer function is

$$G(s) = \frac{V_C(s)}{V(s)} = \frac{1}{RCs + 1}$$

2 Consider a thermocouple which has a transfer function linking its voltage output V and temperature input of

$$G(s) = \frac{30 \times 10^{-6}}{10s + 1}\ \text{V/°C}$$

Determine the response of the system when subject to a step input of size 100°C and hence the time taken to reach 95% of the steady-state value.

Since the transform of the output is equal to the product of the transfer function and the transform of the input, then

$$V(s) = G(s) \times \text{input}(s)$$

A step input of size 100°C, i.e. the temperature of the thermocouple is abruptly increased by 100°C, is $100/s$. Thus

$$V(s) = \frac{30 \times 10^{-6}}{10s + 1} \times \frac{100}{s} = \frac{30 \times 10^{-4}}{10s(s + 0.1)}$$

$$= 30 \times 10^{-4}\frac{0.1}{s(s + 0.1)}$$

The fraction element is of the form $a/s(s + a)$ and so the inverse transform is

$$V = 30 \times 10^{-4}(1 \times e^{-0.1t})\ \text{V}$$

The final value, i.e. the steady-state value, is when $t \to \infty$ and so is when the exponential term is zero. The final value is therefore 30×10^{-4} V. Thus the time taken to reach, say, 95% of this is given by

$$0.95 \times 30 \times 10^{-4} = 30 \times 10^{-4}(1 \times e^{-0.1t})$$

Thus $0.05 = e^{-0.1t}$ and $\ln 0.05 = -0.1t$. The time is thus 30 s.

The percentage overshoot is given by

$$\text{percentage overshoot} = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100\%$$

$$= \exp\left(\frac{-0.44\pi}{\sqrt{1-0.44^2}}\right) \times 100\%$$

The percentage overshoot is thus 21%. The 2% settling time is given by

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.44 \times 2.0} = 4.5 \text{ s}$$

The number of oscillations occurring within the 2% settling time is given by

$$\text{number of oscillations} = \frac{2}{\pi}\sqrt{\frac{1}{\zeta^2} - 1} = \frac{2}{\pi}\sqrt{\frac{1}{0.44^2} - 1} = 1.3$$

19.6 System identification

In Chapters 17 and 18 models were devised for systems by considering them to be made up of simple elements. An alternative way of developing a model for a real system is to use tests to determine its response to some input, e.g. a step input, and then find the model that fits the response. This process of determining a mathematical model is known as **system identification.** Thus if we obtain a response to a step input of the form shown in Figure 19.5 then we might assume that it is a first-order system and determine the time constant from the response curve. For example, suppose the response takes a time of 1.5 s to reach 0.63 of its final height and the final height of the signal is five times the size of the step input. Table 19.1 indicates a time constant of 1.5 s and so the differential equation describing the model is

$$1.5\frac{dx}{dt} + x = 5y$$

An under-damped second-order system will give a response to a step input of the form shown in Figure 19.12. The damping ratio can be determined from measurements of the first and second overshoots with the ratio of these overshoots, i.e. the subsidence ratio, giving the damping ratio. The natural frequency can be determined from the time between successive overshoots. We can then use these values to determine the constants in the second-order differential equation.

Summary

The **natural response** of a system is when there is no input to the system forcing the variable to change but it is just changing naturally. The **forced response** of a system is when there is an input to the system forcing it to change.

A first-order system with no forcing input has a differential equation of the form

$$a_1\frac{dx}{dt} + a_0x = 0$$

and this has the solution $x = e^{-a_0t/a_1}$.

2 Determine the magnitude and phase of the output from a system when subject to a sinusoidal input of $2 \sin(3t + 60°)$ if it has a transfer function of

$$G(s) = \frac{4}{s + 1}$$

The frequency-response function is obtained by replacing s by $j\omega$. Thus

$$G(j\omega) = \frac{4}{j\omega + 1}$$

Multiplying the top and bottom of the equation by $(-j\omega + 1)$,

$$G(j\omega) = \frac{-j4\omega + 4}{\omega^2 + 1} = \frac{4}{\omega^2 + 1} - j\frac{4\omega}{\omega^2 + 1}$$

The magnitude is thus

$$|G(j\omega)| = \sqrt{x^2 + y^2} = \sqrt{\frac{4^2}{(\omega^2 + 1)^2} + \frac{4^2\omega^2}{(\omega^2 + 1)^2}} = \frac{4}{\sqrt{\omega^2 + 1}}$$

and the phase angle is given by $\tan\phi = y/x$ and so

$$\tan\phi = -\omega$$

For the specified input we have $\omega = 3$ rad/s. The magnitude is thus

$$|G(j\omega)| = \frac{4}{\sqrt{3^2 + 1}} = 1.3$$

and the phase is given by $\tan\phi = -3$. Thus $\phi = -72°$. This is the phase angle between the input and the output. Thus the output is $2.6 \sin(3t - 12°)$.

21.3.2 Frequency response for a second-order system

Consider a second-order system with the transfer function (see Section 20.3)

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where ω_n is the natural angular frequency and ζ the damping ratio. The frequency-response function is obtained by replacing s by $j\omega$. Thus

$$G(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega\omega_n + \omega_n^2} = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\zeta\omega_n}$$

$$= \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + j2\zeta\left(\frac{\omega}{\omega_n}\right)}$$

Multiplying the top and bottom of the expression by

$$\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] - j2\zeta\left(\frac{\omega}{\omega_n}\right)$$